Logarithmic Sobolev Inequalities on Homogeneous Spaces

Liangbing Luo, Queen's University

BIRS Workshop, Stochastics and Geometry

September 10, 2024

This is based on

- a joint work with M. Gordina, Logarithmic Sobolev inequalities on non-isotropic Heisenberg groups, in *Journal of Functional Analysis, 2022*
- a joint work with M. Gordina, Logarithmic Sobolev inequalities on homogeneous spaces, Accepted by *IMRN*, 2023, https://arxiv.org/abs/2310.13470

Outline

- Logarithmic Sobolev inequality on $\mathbb{R}^n$
- Hypoelliptic Logarithmic Sobolev inequalities on homogeneous spaces
- Examples
- Further directions

Logarithmic Sobolev Inequality (L. Gross 1975)

$\mathbb{R}^n$

μ: Gaussian measure on $\mathbb{R}^n$ and $d\mu(x) = \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{\frac{n}{2}}} dx$

f: a $C_c^\infty(\mathbb{R}^n)$ function with $\|f\|_{L^2(\mathbb{R}^n, d\mu)} = 1$

$$\int_{\mathbb{R}^n} f^2 \log f^2 d\mu \leqslant C \int_{\mathbb{R}^n} |\nabla f|^2 d\mu$$

where C is independent of n.

Sub-Riemannian manifolds

M: an n-dimensional connected smooth manifold

$\mathcal{H}$: a $C^\infty(M)$-submodule of $\mathrm{Vec}(M)$ satisfying Hörmander's condition, i.e.

$$\mathrm{Lie}(\mathcal{H}_p) = T_pM \text{ for any } p \in M$$

$\langle\cdot,\cdot\rangle_{\mathcal{H}}$: an inner product on $\mathcal{H}$

$(M, \mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$: a sub-Riemannian manifold

$\Delta_{\mathcal{H}}$: sub-Laplacian

Hörmander's condition $\Rightarrow \Delta_{\mathcal{H}}$ is hypoelliptic!

Examples:

- Standard/Isotropic Heisenberg group $\mathbb{H}^1_{\omega_0}$
- $SU(2)$

Hypoelliptic logarithmic Sobolev Inequality on M

$(M, \mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$: a sub-Riemannian manifold

μ: **an analogue of Gaussian measure on M???**

f: a $C_c^\infty(M)$ function with $\|f\|_{L^2(M,d\mu)} = 1$

$$\int_M f^2 \log f^2 d\mu \leqslant C \int_M |\nabla_{\mathcal{H}} f|_{\mathcal{H}} d\mu$$

Is C independent of the dimension?

Fundamental Difficulties

- No canonical choice of reference measure
- The generator $\frac{1}{2}\Delta_{\mathcal{H}}$ is not elliptic but hypoelliptic
- Geometric techniques are not readily available
- Functional inequalities are more difficult to prove or have to be modified

Heisenberg groups:

- H-Q. Li (2006): 3-dimensional, isotropic
- N. Eldredge (2009), W. Hebisch and B. Zegarliński (2010): for arbitrary n
- R. Frank and E. Lieb (2012): on CR sphere
- F. Baudoin and M. Bonnefont (2012): invariant measure
- M. Bonnefont, D. Chafaï and R. Herry (2018): random walk approximation for $n = 1$

- E. Bou Dagher and B. Zegarlinski (2021): an example of non-isotropic case
- Y. Zhang (2021): non-isotropic

Remark

No known results giving with the constant ***independent*** *of the dimension!*

$SU(2)$:

- P. Ługiewicz, B. Zegarlinski (2007)

Main result

Theorem: (Gordina and L. 2023)

M: a homogeneous G-space
G: an n-dimensional connected Lie group
$(G, \mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$: a *left-invariant* sub-Riemannian structure
μ_t^G: heat kernel measure on G of $\Delta_{\mathcal{H}}^G$
$LSI(C(G,t), \mu_t^G)$ holds

Then,

- M has a natural sub-Riemannian structure $(M, \mathcal{H}^M, \langle \cdot, \cdot \rangle_{\mathcal{H}}^M)$ induced by the transitive action of G.

Main result

Theorem: (Gordina and L. 2023)

M: a homogeneous G-space
G: an n-dimensional connected Lie group
$(G, \mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$: a *left-invariant* sub-Riemannian structure
μ_t^G: heat kernel measure on G of $\Delta_{\mathcal{H}}^G$
$LSI(C(G,t), \mu_t^G)$ holds

Then,

- There exists the *unique* hypoelliptic heat kernel measure μ_t^M on M such that the heat equation holds.

Main result

Theorem: (Gordina and L. 2023)

M: a homogeneous G-space
G: an n-dimensional connected Lie group
$(G, \mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$: a *left-invariant* sub-Riemannian structure
μ_t^G: heat kernel measure on G of $\Delta_{\mathcal{H}}^G$
$LSI(C(G,t), \mu_t^G)$ holds

Then,

- $LSI_C(M, \mathcal{H}^M, \mu_t^M)$ holds and

$$C(M,t) = C(G,t).$$

Idea of proof

Homogeneous space characterization theorem

$$M \cong G_p \backslash G$$

$$\Updownarrow$$

$M \cong H \backslash G$ for some closed subgroup H of G.

Type I: Heisenberg-related examples

M: a step-2 sub-Riemannian manifold
$G = \mathbb{H}^1_{\omega_0} \times \cdots \times \mathbb{H}^1_{\omega_0}$: product of 3-dimensional Heisenberg groups
H: a closed subgroup of G $\Rightarrow$ It has a characterization!

Theorem (M. Gordina and L, 2023)

- *M satisfies $LSI(C(M,t), \mu_t^M)$.*
- $C(M,t) = C(\mathbb{H}^1_{\omega_0}, t) = (c(\omega_0))^2\, t.$
- *$C(M,t)$ is independent of its dimension!*

Example 1: Grushin Plane

$M \cong \mathbb{R}^2$: the Grushin plane

$\Delta_{\mathcal{H}}^{M} = \frac{\partial^2}{\partial u^2} + u^2 \frac{\partial^2}{\partial v^2}$: the Grushin operator

$G = \mathbb{H}^1_{\omega_0}$

$H = \{(0, y, 0) : y \in \mathbb{R}\}$

Proposition (M. Gordina and L, 2023)

- *The Grushin plane satisfies* $LSI\left(C(M,t), \mu_t^M\right)$.
- $C(M,t) = C\left(\mathbb{H}^1_{\omega_0}, t\right)$.

Example 2:

Non-isotropic Heisenberg groups G

(V, ω): a finite-dimensional symplectic space

G can be regarded as $V \times \mathbb{R}$ with a non-commutative group law given by

$$(\mathrm{v}, z) \cdot (\mathrm{v}', z') := \left(\mathrm{v} + \mathrm{v}', z + z' + \frac{1}{2}\omega(\mathrm{v}, \mathrm{v}') \right),$$

$(\mathrm{v}, z), (\mathrm{v}', z') \in V \times \mathbb{R}$,

$\omega : V \times V \longrightarrow \mathbb{R}$ is a symplectic bilinear form on V

Non-isotropic Heisenberg groups $\mathbb{H}^n_\omega$

$\mathbb{R}^{2n+1} \cong \mathbb{R}^{2n} \times \mathbb{R}$ equipped with a non-commutative law

$$(\mathrm{v}, z) \cdot (\mathrm{v}', z') := \left(\mathrm{v} + \mathrm{v}', z + z' + \frac{1}{2}\omega(\mathrm{v}, \mathrm{v}') \right),$$

$$(\mathrm{v}, z) = (x_1, y_1, \ldots, x_n, y_n, z) \in \mathbb{R}^{2n} \times \mathbb{R}$$

$$(\mathrm{v}', z') = (x_1', y_1', \ldots, x_n', y_n', z') \in \mathbb{R}^{2n} \times \mathbb{R}$$

$\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ is a symplectic bilinear form

We have $\omega(\mathrm{v}, \mathrm{v}') = \sum_{i=1}^n \alpha_i (x_i y_i' - x_i' y_i)$, $\alpha_i \in \mathbb{R}^+$.

LSI on non-isotropic Heisenberg groups

$G = \mathbb{H}^1_{\omega_0} \times \cdots \times \mathbb{H}^1_{\omega_0}$

$H = \{(0, z_1, \cdots, 0, z_n) \in \mathbb{H}^1_{\omega_0} \times \cdots \times \mathbb{H}^1_{\omega_0} : \sum_{i=1}^n \alpha_i z_i = 0\}$

Theorem (M. Gordina and L,2022)

- $\mathbb{H}^n_\omega$ *satisfies* $LSI(C(\mathbb{H}^n_\omega, t), \mu^\omega_t)$.
- $C(\mathbb{H}^n_\omega, t) = C(\mathbb{H}^1_{\omega_0}, t)$.
- *The logarithmic Sobolev constant* $C(\mathbb{H}^n_\omega, t)$ *is independent of the symplectic bilinear form* ω *and the dimension!*

Infinite-dimensional Heisenberg-type groups G

(W, H, μ): a real abstract Wiener space
G can be regarded as $W \times \mathbb{R}$ with a non-commutative group law given by

$$(w_1, c_1) \cdot (w_2, c_2) = \left(w_1 + w_2, c_1 + c_2 + \frac{1}{2}\omega(w_1, w_2) \right),$$

$$(w_i, c_i) \in W \times \mathbb{R}, \; i = 1, 2,$$

$$\omega : W \times W \to \mathbb{R}$$

is a continuous surjective anti-symmetric bilinear form on W.

Remark

$G_{CM} := H \times \mathbb{R}$ is the Cameron-Martin subgroup of G.

No Haar measrue!

g_t: Brownian motion whose generator is $\frac{1}{2}L$

μ_t: the law of g_t for $t > 0$

Theorem (M. Gordina and L, 2022)

There exists a Markovian closed symmetric form (Dirichlet form) $\mathcal{E}_t(\cdot,\cdot)$ associated to the heat kernel measure μ_t on $L^2(G, d\mu_t)$.

Logarithmic Sobolev inequalities on G

Theorem (M. Gordina and L, 2022)

For $f \in \mathcal{D}(\mathcal{E}_t)$ *and* $t > 0$,

$$\int_G f^2 \log f^2 d\mu_t - \left(\int_G f^2 d\mu_t\right) \log\left(\int_G f^2 d\mu_t\right) \leqslant C(\omega, t)\mathcal{E}_t(f, f).$$

Logarithmic Sobolev inequalities on G

Theorem (M. Gordina and L, 2022)

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$$\int_G f^2 \log f^2 d\mu_t - \left(\int_G f^2 d\mu_t\right) \log \left(\int_G f^2 d\mu_t\right) \leqslant C(\mathbb{H}^1_{\omega_0}, t)\mathcal{E}_t(f, f).$$

Example 3: Compact Heisenberg nilmanifolds

M = a compact Heisenberg nilmanifold
G = $(2n+1)$-dimensional isotropic Heisenberg group $\mathbb{H}^n$
H = a lattice subgroup of G

Theorem (M. Gordina and L,2023)

- *M satisfies $LSI(C(M,t), \mu_t^M)$.*
- *$C(M,t) = C\left(\mathbb{H}^1_{\omega_0}, t\right)$.*
- *The logarithmic Sobolev constant $C(M,t)$ is independent of the dimension!*

Type II: $SU(2)$-related examples

M: a sub-Riemannian manifold
$G = SU(2) \times \cdots \times SU(2)$
H: a closed subgroup of G

Theorem (Gordina and L, 2023)

- *M satisfies LSI* $\left(C\left(M, t\right), \mu_t^M\right)$.
- $C\left(M, t\right) = C\left(SU(2), t\right)$.
- $C\left(M, t\right)$ *is independent of its dimension!*

Example 1: Hopf fibration

$M = \mathbb{CP}^1$

$G = SU(2)$

$H = S^1$

Proposition (Gordina and L, 2023)

- $\mathbb{CP}^1$ *satisfies LSI* $\left(C\left(\mathbb{CP}^1, t\right), \mu_t^{\mathbb{CP}^1} \right)$.
- $C\left(\mathbb{CP}^1, t\right) = C\left(SU(2), t\right)$.

Example 2: $SO(3)$

$M = SO(3)$

$G = SU(2)$

$H = \mathbb{Z}_2$

Theorem (Gordina and L, 2023)

- $SO(3)$ *satisfies LSI* $\left(C\left(SO(3), t\right), \mu_t^{SO(3)} \right)$.
- $C\left(SO(3), t\right) = C\left(SU(2), t\right)$.

Example 3: $SO(4)$

$M = SO(4)$

$G = SU(2) \times SU(2)$

$H = \mathbb{Z}_2$

Theorem (Gordina and L, 2023)

- $SO(4)$ *satisfies LSI* $\left(C\left(SO(4), t\right), \mu_t^{SO(4)} \right)$.
- $C\left(SO(4), t\right) = C\left(SU(2), t\right)$.
- $C\left(SO(4), t\right)$ *is independent of its dimension!*

Further Directions

- Other groups? Other spaces? Other symmetries?
- Infinite-dimensional extensions?
- Related problems?

Thank you!